Exercise 23

Sketch the graph of a function f for which $f(0) = 0$, $f'(0) = 3$, $f'(1) = 0$, and $f'(2) = -1$.

Solution

There are four conditions to be satisfied, so there are four constants to be determined in the unknown function. Let the function be a cubic polynomial for simplicity.

$$f(x) = Ax^3 + Bx^2 + Cx + D$$

Take the derivative of $f(x)$.

$$f'(x) = 3Ax^2 + 2Bx + C$$

Now apply the given conditions to obtain a system of four equations for the four unknowns.

$$\begin{cases} f(0) = A(0)^3 + B(0)^2 + C(0) + D = 0 \\ f'(0) = 3A(0)^2 + 2B(0) + C = 3 \\ f'(1) = 3A(1)^2 + 2B(1) + C = 0 \\ f'(2) = 3A(2)^2 + 2B(2) + C = -1 \end{cases}$$

The first two equations give $C = 3$ and $D = 0$. As a result, the last two equations become

$$\begin{cases} 3A + 2B + 3 = 0 \\ 12A + 4B + 3 = -1 \end{cases}$$

$$\begin{cases} 3A + 2B = -3 \\ 12A + 4B = -4 \end{cases}.$$

Multiply both sides of the first equation by -2

$$\begin{cases} -6A - 4B = 6 \\ 12A + 4B = -4 \end{cases}$$

and then add the respective sides to eliminate B.

$$6A = 2$$

Solve for A.

$$A = \frac{1}{3}$$

To get B, multiply both sides of the first equation instead by -4

$$\begin{cases} -12A - 8B = 12 \\ 12A + 4B = -4 \end{cases}$$

and then add the respective sides to eliminate A.

$$-4B = 8$$

Solve for B.

$$B = -2$$

Now that A, B, C, and D are known, the function is known and can be plotted versus x.

$$f(x) = \frac{1}{3}x^3 - 2x^2 + 3x$$